Poisson's Eq Via FEM

$$\nabla^2 \Phi = -\rho/\varepsilon$$

Functional: $$I(\Phi) = \frac{1}{2}\int_D \varepsilon |\nabla \Phi|^2 - 2\rho\Phi \, dV$$

Let: $$\Phi^{(e)}(x,y) = \underline{\phi}^T \underline{\alpha} = \underline{\alpha}^T \underline{\phi} = \sum_{i=1}^{3} \phi_i \alpha_i(x,y)$$

$$\rho^{(e)}(x,y) = \underline{\rho}^T \underline{\alpha} = \underline{\alpha}^T \underline{\rho} = \sum_{i=1}^{3} \rho_i \alpha_i(x,y)$$

$\alpha_i(x,y)$ are the shape functions

$$I^{(e)}(\Phi) = \frac{1}{2}\int_\Delta \varepsilon^{(e)} \underline{\phi}^T (\nabla\underline{\alpha} \cdot \nabla\underline{\alpha}^T)\underline{\phi} - 2\underline{\phi}^T \underline{\alpha}\,\underline{\alpha}^T \underline{\rho} \, dV$$

$$= \frac{1}{2}\varepsilon^{(e)} \underline{\phi}^T \int_\Delta \nabla\underline{\alpha} \cdot \nabla\underline{\alpha}^T \, dx\,dy \, \underline{\phi} - \underline{\phi}^T \int_\Delta \underline{\alpha}\,\underline{\alpha}^T dx\,dy\, \underline{\rho}$$

let $$S = \int_\Delta \nabla\underline{\alpha} \cdot \nabla\alpha^T \, dx\,dy \qquad T = \int_\Delta \underline{\alpha}\,\underline{\alpha}^T \, dx\,dy.$$

minimizing $I(\underline{\phi})$ w.r.t. $\underline{\phi}$ we set:

$$\frac{\partial I}{\partial \underline{\phi}} = 0 = \varepsilon^{(e)} S \underline{\phi} - \frac{\partial}{\partial \underline{\phi}} \left(\underline{\phi}^T T \underline{\rho} \right)$$

$$= \varepsilon^{(e)} S \underline{\phi} - T \underline{\rho} = 0$$

$$T = \int_D \underline{\alpha} \cdot \underline{\alpha}^T dv = \frac{Ae}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

General Wave Equation:

$$\nabla^2 \underline{\Phi} + k^2 \underline{\Phi} = g.$$

$$I(\underline{\Phi}) = \frac{1}{2} \int_D |\nabla \underline{\Phi}|^2 - k^2 \underline{\Phi}^2 + 2 \underline{\Phi} g \, dV$$

$$\underline{I}^{(e)}(\underline{\Phi}) = \frac{1}{2} \underline{\Phi}^T S \underline{\Phi} - \frac{k^2}{2} \underline{\Phi}^T T \underline{\Phi} + \underline{\Phi}^T T \underline{g}$$